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Stable and Unstable Manifolds of the Hénon Mapping

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By using a parametric representation of the stable and unstable manifolds, we prove that for some given values of the parameter (in particular in the case first investigated by Hénon) the Hénon mapping has a transversal homoclinic orbit.

KEY WORDS: Hénon mapping; strange attractor; stable and unstable manifolds; homoclinic point.

1. INTRODUCTION

Since Hénon's analysis,[1] considerable interest has been devoted to the mapping of the plane into itself:

$$T(x, y) = (y + 1 - ax^2, bx) \tag{1.1}$$

Hénon studied T in the case $a = 1.4$, $b = 0.3$; numerical investigations of (1.1)[1-3] have shown that for these and other values of the parameters the system (1.1) seems to exhibit a "chaotic behavior." This behavior was related via a theorem by Smale, to the existence of a transversal homoclinic orbit.[3] Smale's theorem[4] (see also Ref. 5) states that if a diffeomorphism F has a transversal homoclinic orbit, then there exists a Cantor set Λ in which, for some M, F^M is topologically equivalent to the shift automorphism. Curry, in Ref. 3, gave numerical evidence to the existence of a transversal homoclinic orbit in the case $a = 1.4$, $b = 0.3$. More recently Marotto[6] proved analytically that for $a > 1.55$ and b small enough such an orbit exists; however, Marotto's proof does not provide an explicit range of b values for which his results hold. Here, by using a parametric representation of the stable and unstable manifolds and with the aid of a

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computer, used as a tool for obtaining rigorous estimates, we prove that a transversal homoclinic orbit exists for $a = 1.4$, $b = 0.3$.

Note: After the completion of this work we have received a paper by Misuriewicz and Szewc[7] in which the existence of a homoclinic orbit for the Hénon mapping is proved, in the case $a = 1.4$, $b = 0.3$, by using computations simpler than those presented here; nevertheless we think it is not useless to present our proof for two reasons: (a) our method allows us to plot the stable and unstable manifolds of the mapping, and this could be of some interest in itself; (b) the proof given in Ref. 7 is based on the existence of a quadrilateral mapped into itself by the Hénon map; such a quadrilateral does not exist if, for example, the parameter a is increased enough, while our method, in principle, works whenever a homoclinic point exists.

2. STABLE AND UNSTABLE MANIFOLDS

Let $P_0 = \{\xi_0, \eta_0)$ be one fixed point of the mapping (1.1). We denote by α_1, α_2 the eigenvalues of the derivative of T at P_0. If, as we suppose, $a > \frac{3}{4}(1-b)^2$ we can choose $|\alpha_1| > 1$, $|\alpha_2| < 1$.

The stable and unstable manifolds of T at P_0 are characterized, by the stable manifold theorem, as the images of two immersions $\gamma_i : R \to R^2$ $(i = 1, 2)$ such that

$$(T \circ \gamma_i)(t) = \gamma_i(\alpha_i t) \tag{2.1}$$

If we put $\gamma_i(t) = (\xi_i(t), \eta_i(t))$, (2.1) can be written

$$\eta_i(t) + 1 - a\xi_i^2(t) = \xi_i(\alpha_i t), \qquad b\xi_i(t) = \eta_i(\alpha_i t) \tag{2.2}$$

Hence we have

$$\eta_i(t) + 1 - ab^{-2}\eta_i^2(\alpha_i t) = b^{-1}\eta_i(\alpha_i^2 t) \tag{2.3}$$

It is known (see for example Ref. 5) that the γ_i's are analytic. We compute explicitly the coefficients of the expansion

$$\eta_i(t) = \eta_0 + \sum_{n=1}^{\infty} c_n^{(i)} t^n \tag{2.4}$$

By substituting (2.4) in (2.3) we get

$$n \geqslant 1 \qquad c_n^{(i)} = ab^{-2}\left(2\eta_0 c_n^{(i)} + \sum_{k=1}^{n-1} c_k^{(i)} c_{n-k}^{(i)}\right)\alpha_i^n + b^{-1}\alpha_i^{2n} c_n^{(i)} \tag{2.5}$$

For $n = 1$ (2.5) gives

$$\alpha_i^2 + 2ab^{-1}\eta_0\alpha_i - b = 0 \tag{2.6}$$

(i.e., the equation defining the eigenvalues α_i). The choice of $c_1^{(i)}$ is arbitrary and fixes the scale of the parameter t; we put $c_1^{(i)} = 1$. Equation (2.5) gives, by recurrence, the other $c_n^{(i)}$. We have

$$n > 1 \qquad c_n^{(i)} = \beta_{n,i} \sum_{k=1}^{n-1} c_k^{(i)} c_{n-k}^{(i)} \tag{2.7}$$

where $\beta_{n,i}$ is defined by

$$\beta_{n,i}^{-1} = -a^{-1} b \alpha_i^n (1 - \alpha_1 \alpha_i^{-n})(1 - \alpha_2 \alpha_i^{-n}) \tag{2.8}$$

From now on we denote by $\eta_i(t)$ the function obtained by substituting (2.7), (2.8) in (2.4). Since $\alpha_1, \alpha_2 \neq 1$ it is easy to check, via an exponential upper bound on the $\beta_{n,i}$, that the radius of convergence ρ of (2.4) is different from zero. On the other hand it is clear that if $|t|, |\alpha_i t|, |\alpha_i^2 t| < \rho$, $\eta_i(t)$ satisfies (2.3); hence $\eta_i(t)$ is an entire function [otherwise one could use (2.3) for extending $\eta_i(t)$ analytically to the whole complex plane].

By computing a suitable number of $c_n^{(i)}$ the expansion (2.4) can be used to get a diagram of the stable and unstable manifolds. A segment of the two manifolds corresponding to the right fixed point of T in the case $a = 1.4$, $b = 0.3$ is represented in Fig. 1.

3. HOMOCLINIC POINTS

The diagram in Fig. 1, obtained truncating the series (2.4) at the term $n = 100$, gives evidence of the existence of homoclinic points in the case $a = 1.4$, $b = 0.3$. In this section we give a proof of this fact.

We consider the unstable and stable manifolds, γ_1 and γ_2, of T at the right fixed point $P_0 = (\xi_0 = 0.63135447\ldots, \eta_0 = 0.18940634\ldots)$. In particular we consider the segments of γ_1 and γ_2 contained between the points $P_1 = \gamma_1(-0.42), P_2 = \gamma_1(-0.41)$, $Q_1 = \gamma_2(-0.61)$, $Q_2 = \gamma_2(-0.57)$. (These segments were empirically chosen as an ansatz from the numerical computations: their intersection, whose existence we are proving, is represented in Fig. 1 by the point O.) We put $t_0 = -0.415$, $I_1 = [-0.42, -0.41]$ and we write

$$\eta_i(t) = \eta_i^{20}(t) + \eta_i^R(t) \tag{3.1}$$

where

$$\eta_i^R(t) = \sum_{n=21}^{\infty} c_n^{(i)} t^n$$

In Appendix A the following bounds are proved:

$$\max_{|t|<1} |\eta_i^R(t)| < 10^{-3}, \qquad \max_{|t|<1} |\eta_i^{R'}(t)| < 2.6 \times 10^{-2}$$

$$\max_{|t|<1} |\eta_i^{R''}(t)| < 1.1, \qquad \max_{|t|<1} |\eta_i^{R'''}(t)| < 63 \tag{3.2}$$

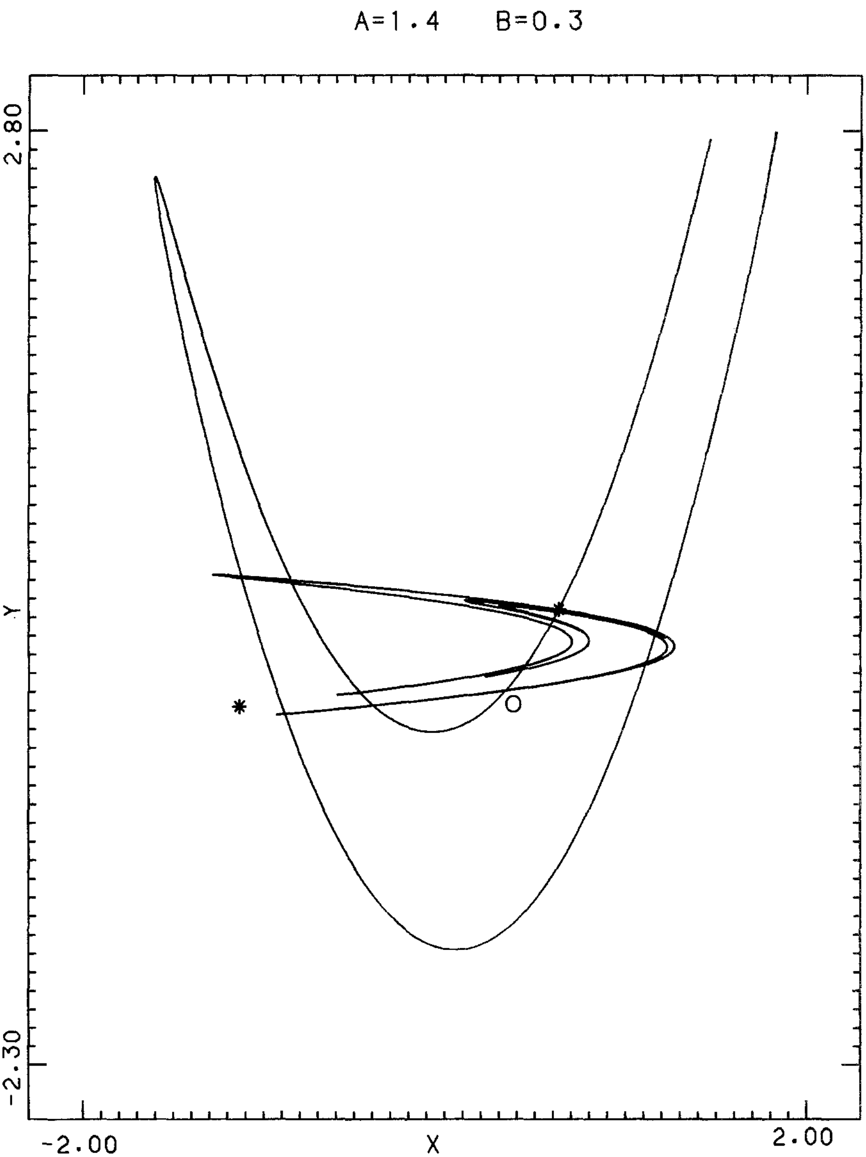

Fig. 1. γ_1 together with $\gamma_2(-27, 2)$ for $a = 1.4$, $b = 0.3$. The two asterisks represent the two fixed points of T. The homoclinic point whose existence is proved in Section 3 is at O.

Analogous inequalities concerning $\xi_i(t)$ can easily be obtained by using (2.2).

On the other hand, explicit computations (made with the aid of the computer, by using the estimates on the computational error given in Appendix B) give

$$\max_{|t|<1} |\eta_1^{20'''}(t)| < \sum_{n=3}^{20} n(n-1)(n-2)|c_n^{(1)}| < 177$$

$$\eta_i^{20'}(t_0) = \sum_{n=1}^{20} n c_n^{(1)}(-0.415)^{n-1} \in (0.857, 0.858)$$

$$\eta_1^{20''}(t_0) = \sum_{n=2}^{20} n(n-1) c_n^{(1)}(-0.415)^{n-2} \in (2.29, 2.30)$$

Taking into account the bounds (3.2) we get

$$\eta_1'(t_0) \in (0.83, 0.89) \tag{3.3}$$

Furthermore, we have

$$\begin{aligned} \forall t \in I_1 \qquad |\eta_1'(t) - \eta_1'(t_0)| &\leqslant 5 \times 10^{-3}\Big[|\eta_1''(t_0)| + 5 \times 10^{-3} \max_{|t|<1} |\eta_1'''(t)|\Big] \\ &< 5 \times 10^{-3}\Big[|\eta_1^{20''}(t_0)| + |\eta_1^{R''}(t_0)| \\ &\qquad + 5 \times 10^{-3}(177 + 63)\Big] \\ &< 5 \times 10^{-3}[2.3 + 1.1 + 5 \times 10^{-3} \times 240] \\ &= 2.3 \times 10^{-2} \end{aligned}$$

Hence we get

$$\forall t \in I_1 \qquad \eta_1'(t) \in (0.80, 0.92) \tag{3.4}$$

Analogous computations show that

$$\begin{aligned} &\forall t \in I_1 \qquad \xi_1'(t) \in (6.3, 6.8) \\ &\forall t \in I_2 \equiv [-0.61, -0.57] \qquad \eta_2'(t) \in (0.45, 0.60), \quad \xi_2'(t) \in (0.3, 0.6) \end{aligned} \tag{3.5}$$

By collecting together (3.3), (3.4), (3.5) we have

$$\begin{aligned} &\forall t \in I_1 \qquad 0.11 < \frac{0.8}{6.8} < \frac{\eta_1'(t)}{\xi_1'(t)} < \frac{0.92}{6.3} < 0.15 \\ &\forall t \in I_2 \qquad 0.75 < \frac{0.45}{0.6} < \frac{\eta_2'(t)}{\xi_2'(t)} < \frac{0.6}{0.3} = 2 \end{aligned} \tag{3.6}$$

The last inequalities show that any intersection between $\gamma_1(I_1)$ and $\gamma_2(I_2)$ must be transversal. In order to prove that such an intersection actually exists, it suffices to prove that

$$\begin{gathered} y(Q_1) < y(P_1) < y(P_2) < y(Q_2) \\ x(P_1) < x(Q_1) < x(Q_2) < x(P_2) \end{gathered} \tag{3.7}$$

The inequalities (3.7), indeed, imply that $\gamma_1(I_1)$ and $\gamma_2(I_2)$ connect different pairs of opposite sides of the rectangle $\Lambda = \{x(P_1) \leqslant x \leqslant x(P_2);\ y(Q_1) \leqslant y \leqslant y(Q_2)\}$; since, by (3.4), (3.5), $\gamma_1(I_1)$ and $\gamma_2(I_2)$ are contained in Λ, they must intersect.

Explicit computations, together with the estimates given in Appendix B, give

$$\begin{aligned} &\xi_1^{20}(-0.42) \in (0.3025, 0.3026), \quad \eta_1^{20}(-0.42) \in (-0.2599, -0.2598) \\ &\xi_1^{20}(-0.41) \in (0.3683, 0.3684), \quad \eta_1^{20}(-0.41) \in (-0.2513, -0.2512) \\ &\xi_2^{20}(-0.61) \in (0.3274, 0.3275), \quad \eta_2^{20}(-0.61) \in (-0.2677, -0.2676) \\ &\xi_2^{20}(-0.57) \in (0.3465, 0.3466), \quad \eta_2^{20}(-0.57) \in (-0.2464, -0.2463) \end{aligned}$$

Taking into account the first of (3.2) and the analogous inequality

$$\max_{|t| < \alpha_i^{-1}} |\xi_i^R(t)| \leqslant \tfrac{1}{3} 10^{-2}$$

we get that the coordinates of P_1, P_2, Q_1, Q_2 actually satisfy (3.7).

4. SOME REMARKS ON THE PICTURES

In order to draw a picture of γ_1 and γ_2, we have computed the series (2.4) truncated at the term $n = 100$ for $|t| < 1$, and we have continued by iterating (2.1) for larger values of t.

This procedure rapidly gives the full picture of the image of γ_1, since its closure, $\overline{\operatorname{Im}\gamma_1}$, is a set of Lebesgue measure zero (since it is a bounded[2] T-invariant measurable set) which, on the scale of the picture, coincides with a segment of γ_1 corresponding to a short interval in t (say $|t| < 20$). In the case $a = 1.4$, $b = 0.3$ (Figs. 1 and 2) $\operatorname{Im}\gamma_1$ seems to coincide with the strange attractor observed by Hénon (it is believed that the attractor actually coincides with $\overline{\operatorname{Im}\gamma_1}$). More generally the attractor appears to be contained in $\overline{\operatorname{Im}\gamma_1}$; in particular for $a = 1.3$, $b = 0.3$, where there is known

[2] In the case $a = 1.4$ this is a consequence of the fact, proved in Ref. 1, that P_0 is internal to a bounded region mapped into itself by T. There is numerical evidence (and it seems not too difficult to prove) that the same statement holds for $a = 1.3$.

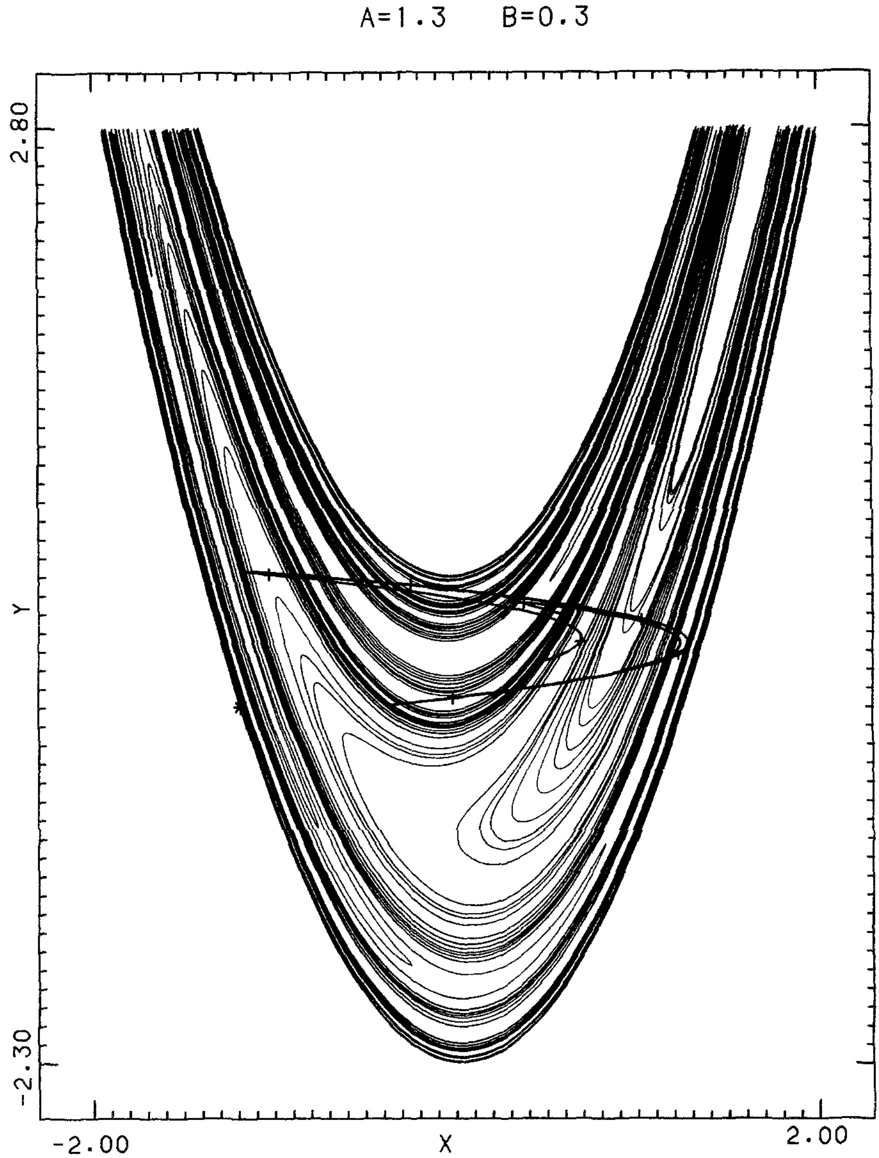

Fig. 2. γ_1 and the intersections of $\gamma_2(-5 \times 10^8, 2)$ with the rectangle $[-2.3 \leqslant x \leqslant 2.8;\ -2 \leqslant y \leqslant 2]$ for $a = 1.4$, $b = 0.3$.

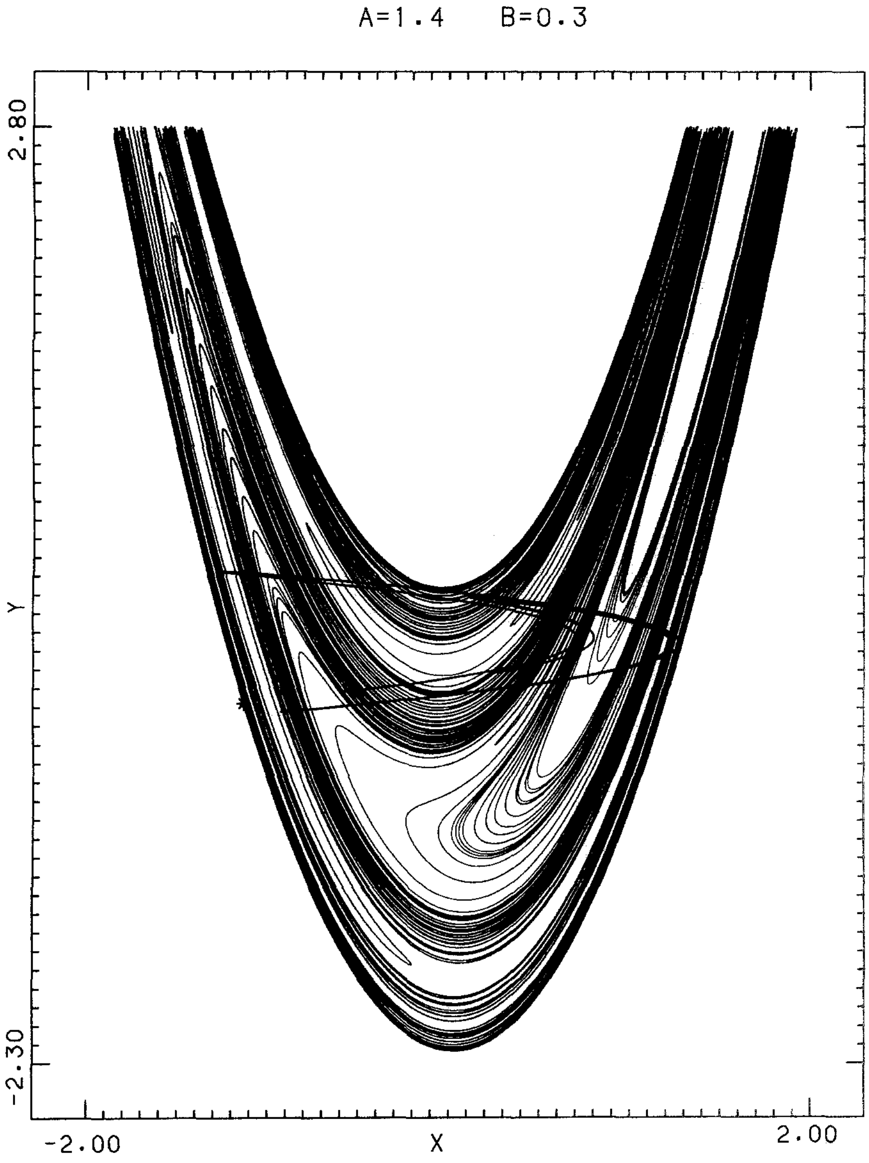

Fig. 3. γ_1 and the intersections of $\gamma_2(-5 \times 10^8, 2)$ with the rectangle $[-2.3 \leqslant x \leqslant 2.8;\ -2 \leqslant y \leqslant 2]$ for $a = 1.3$, $b = 0.3$. The crosses indicate the seven points of the stable cycle.

to exist an attracting cycle of period seven, the points of this cycle (marked in Fig. 3 with a cross) belong to $\overline{\operatorname{Im}\gamma_1}$.

We remark that the existence of an open set in the plane which is attracted by $\overline{\operatorname{Im}\gamma_1}$ is an almost obvious consequence of the existence of homoclinic points of T. Consider, indeed, the region Λ of the plane which is bounded by γ_1^a and γ_2^a, where γ_1^a $[\gamma_2^a]$ is the segment of γ_1 $[\gamma_2]$ contained between P_0 and a given homoclinic point O. If $P \in \Lambda$, $T^kP \in T^k\Lambda$ and the distance between T^kP and the boundary $\delta T^k\Lambda$ of $T^k\Lambda$ goes to zero as k goes to infinity (since the Jacobian of T is constant and less than unity). On the other hand, for $k > 0$,

$$\delta T^k\Lambda = T^k\gamma_1^a \cup T^k\gamma_2^a \subset \gamma_1 \cup T^k\gamma_2^a$$

since the length of $T^k\gamma_2^a$ goes to zero as k goes to infinity, the previous statement follows.

The picture of $\operatorname{Im}\gamma_2$ is more difficult to obtain. For positive values of t γ_2 rapidly diverges, whereas $\gamma_2(-\infty, 0)$ appears to be dense in an unbounded region of positive Lebesgue measure. In Figs. 2 and 3 we have drawn the intersection of $\gamma_2(-5 \times 10^8, 2)$ with a rectangle R containing $\operatorname{Im}\gamma_1$. These pictures are somewhat difficult to obtain because most "returns" of γ_2 in R occur in very short t intervals (of length even less than 0.2) whereas for other values of t in the interval $[-5 \times 10^8, 2]$ the coordinates of γ_2 assume values exceeding the capacity of the computer.

Figure 2 is compatible with the conjecture that $\overline{\operatorname{Im}\gamma_2}$ coincides with the basin of attraction of $\overline{\operatorname{Im}\gamma_1}$, whereas in Figure 3 some white regions containing the points of the attracting cycle are visible.

ACKNOWLEDGMENTS

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APPENDIX A

First we prove the bound

$$\forall n \geqslant 6 \qquad |\beta_{n,1}| < \tfrac{14}{3}(1.9)^{-n} \tag{A.1}$$

We have

$$\begin{aligned} |\beta_{n,1}| &< ab^{-1}|\alpha_1|^{-n}\left(1 - |\alpha_1|^{-n+1}\right)^{-1}\left(1 - |\alpha_2|\,|\alpha_1|^{-n}\right)^{-1} \\ &< ab^{-1}|\alpha_1|^{-n}\left(1 - |\alpha_1|^{-5}\right)^{-1}\left(1 - |\alpha_2|\,|\alpha_1|^{-6}\right)^{-1} \end{aligned}$$

Since $a = 1.4$, $b = 0.3$, $\alpha_1 = -1.9237\ldots$, $\alpha_2 = 0.1559\ldots$ we get

$$|\beta_{n,1}| < \tfrac{14}{3}(1.9)^{-n}(1.92/1.9)^{-6}\left[1-(1.9)^{-5}\right]^{-1}\left[1-0.16(1.9)^{-6}\right]^{-1} < \tfrac{14}{3}(1.9)^{-n}$$

Now we prove the following upper bound on the $c_n^{(1)}$:

$$\forall n \geqslant 1 \qquad |c_n^{(1)}| < \min\{1.6, 12n(1.9)^{-n}\} \tag{A.2}$$

In the case $n \leqslant 5$ one can directly verify that $|c_n^{(1)}| < 1.6 < 12n(1.9)^{-n}$. For $n \geqslant 6$, by using (A.1), (A.2) can be proved by recurrence. We have

$$|c_n^{(1)}| \leqslant |\beta_{n,1}| \sum_{k=1}^{n-1} |c_k^{(1)}||c_{n-k}^{(1)}| \leqslant \tfrac{14}{3}(1.9)^{-n}(n-1)(1.6)^2 < 12n(1.9)^{-n} < 1.6$$

By using (A.2) we have

$$\max_{|t|<1} |\eta_1^R(t)| < \sum_{n=21}^{\infty} |c_n^{(1)}| < \sum_{n=21}^{\infty} 12n(1.9)^{-n}$$

Since, if $n > 20$, $n(1.9)^{-n} < (1.64)^{-n}$ we get

$$\max_{|t|<1} |\eta_1^R(t)| < \sum_{n=21}^{\infty} 12(1.64)^{-n} = 12(1.64)^{-21}\frac{41}{16} < 10^{-3}$$

In a analogous way we have

$$\max_{|t|<1} |\eta_1^{R'}(t)| < \sum_{n=21}^{\infty} n|c_n^{(1)}| < 12\sum_{n=21}^{\infty} n^2(1.9)^{-n} < 12\sum_{n=21}^{\infty}(1.42)^{-n} < 2.6 \times 10^{-2}$$

$$\max_{|t|<1} |\eta_1^{R''}(t)| < \sum_{n=21}^{\infty} n^2|c_n^{(1)}| < 12\sum_{n=21}^{\infty} n^3(1.9)^{-n} < 12\sum_{n=21}^{\infty}(1.22)^{-n} < 1.1$$

$$\max_{|t|<1} |\eta_1^{R'''}(t)| < \sum_{n=21}^{\infty} n^3|c_n^{(1)}| < 12\sum_{n=21}^{\infty} n^4(1.9)^{-n} < 12\sum_{n=21}^{\infty}(1.06)^{-n} < 63$$

Since $|\alpha_2| < |\alpha_1|$, the same inequalities hold, a fortiori, for $n_2^R(t)$.

APPENDIX B

In this appendix we give an estimate on the computational error made by the computer in calculating the polynomial $\eta_1^{20}(t)$. For this purpose we premise some remarks:

(a) All the computations have been performed on a CDC 7600. In this computer a floating point number is represented by a 60-bit word: 48 bits for the integer coefficient, 11 bits for the exponent, and one for the sign.

Furthermore, the machine instructions for multiplication, division, addition of numbers of the same sign, whenever the operands are normalized, deliver normalized results with a relative error less than or equal to $u = 2^{-47}$, provided that neither overflow (i.e., an exponent greater than 2^{10}) nor underflow (i.e., an exponent less than -2^{10}) occurs.

(b) We have programmed in FORTRAN language. It is possible, by a suitable arrangement in the programming, to obtain that the addition of normalized operands of opposite sign also provides a normalized result with a relative error less than or equal to u. Hence, taking into account that the initial data are memorized in normalized form and that neither overflow nor underflow can occur,[3] we are sure that each elementary arithmetic operation produces a relative error of no more than u.

(c) Our FORTRAN program includes only elementary arithmetic operations (except for the computation of the $\tilde{\alpha}_i$, which will be considered later); in particular, any exponentiation is performed through a sequence of multiplications. A conveniently ordered sequence of FORTRAN instructions and an appropriate use of parentheses in the algebraic expressions assure that the sequence of machine language operations is the desired one.

(d) The eigenvalues α_1 and α_2 ($\alpha_1 < 0, \alpha_2 > 0$) have been represented in the computer by two numbers $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, obtained by using double precision and, then, by truncating the 96-bit expansion of the coefficients to the 48 most significant bits. Elementary computations show that the α_i can be obtained as solutions of the equation

$$f(\alpha) = (3/\alpha - 10\alpha + 7)^2 = 609$$

Then, in order to verify that the relative errors $|\alpha_i - \tilde{\alpha}_i|/\alpha_i$ are less than u, it suffices to check the inequalities

$$\begin{gathered} f(\tilde{\alpha}_1) < 609 < f[\tilde{\alpha}_1(1+u)] \\ f[\tilde{\alpha}_2(1+u)] < 609 < f(\tilde{\alpha}_2) \end{gathered} \tag{B.1}$$

Indeed, since $f(\alpha)$ decreases in the regions considered, inequalities (B.1) imply

$$\begin{gathered} \tilde{\alpha}_1(1+u) < \alpha_1 < \tilde{\alpha}_1 \\ \tilde{\alpha}_2 < \alpha_2 < \tilde{\alpha}_2(1+u) \end{gathered}$$

Since $\alpha_1 < 0$, $\alpha_2 > 0$, we get

$$\begin{gathered} \alpha_1 < \tilde{\alpha}_1 < \frac{\alpha_1}{1+u} < \alpha_1(1-u) \\ \alpha_2(1-u) < \frac{\alpha_2}{1+u} < \tilde{\alpha}_2 < \alpha_2 \end{gathered} \tag{B.2}$$

[3] This can easily be verified considering the numerical values of the $\beta_{n,i}$ and $c_n^{(i)}$ for $i = 1, 2$, $n \leqslant 20$; in particular, $10^{-43} < |c_n^{(i)}| < 2$.

The inequalities (B.1) have been checked by using double precision. Since all the values obtained by the computer for $f(\tilde{\alpha}_i)$, $f[\tilde{\alpha}_i(1+u)]$, differ from 609 more than $2^{-51}\times 609$, the errors in calculating the function f are clearly irrelevant to our purpose.

The inequalities (B.2) and the above remarks imply that the $\beta_{n,i}$ are represented in the computer by some numbers $\tilde{\beta}_{n,i}$ given by

$$\tilde{\beta}_{n,1} = -\tfrac{14}{3}(1+\eta_1)^{2n+5}\alpha_1^{-n}\Big\{\Big[1-\alpha_1^{-n+1}(1+\eta_2)^{2(n-1)}\Big] \times\Big[1-\alpha_2\alpha_1^{-n}(1+\eta_3)^{2n+1}\Big]\Big\}^{-1}$$

where $|\eta_i| < u$ $(i=1,2,3)$. We have

$$\frac{\tilde{\beta}_{n,1}}{\beta_{n,1}} = \big[1+(2n+6)\delta_1\big]\left[1-\frac{\alpha_1^{-n+1}}{1-\alpha_1^{-n+1}}(2n-1)\delta_2\right] \times\left[1-\frac{\alpha_2\alpha_1^{-n}}{1-\alpha_2\alpha_1^{-n}}2(n+1)\delta_3\right]$$

where $|\delta_i| < u$ $(i=1,2,3)$. Since $|\alpha_1| > 1.9$, $|\alpha_2| < 0.16$, we have (for $n \geqslant 2$)

$$\left|\frac{\alpha_1^{-n+1}}{1-\alpha_1^{-n+1}}(2n-1)\right| < 1.5, \qquad \left|\frac{\alpha_2\alpha_1^{-n}}{1-\alpha_2\alpha_1^{-n}}(2n+2)\right| < 0.5$$

By using the bounds above we get

$$\left|\frac{\tilde{\beta}_{n,1}-\beta_{n,1}}{\beta_{n,1}}\right| \leqslant (2n+9)u \tag{B.3}$$

We put $\epsilon_n = \tilde{c}_n^{(1)} - c_n^{(1)}$, where $\tilde{c}_n^{(1)}$ is the representation used by the computer for $c_n^{(1)}$. Since $c_1^{(1)} = 1$, $|c_2^{(1)}| = |\beta_{2,1}| < 0.87$, we have $\epsilon_1 = 0$, $\epsilon_2 < 13|\beta_{2,1}|u < 12u$. For $n \geqslant 2$ we have

$$|\epsilon_n| \leqslant |c_n^{(1)}|\left(1+\left|\frac{\tilde{\beta}_{n,1}-\beta_{n,1}}{\beta_{n,1}}\right|\right)u + \tilde{\beta}_{n,1}\left[(n-2)u + u\sum_{k=1}^{n-1}|\tilde{c}_k^{(1)}\tilde{c}_{n-k}^{(1)}| + \sum_{k=1}^{n-1}\left(|\epsilon_k|\,|\tilde{c}_{n-k}^{(1)}| + |\epsilon_{n-k}|\,|c_k^{(1)}|\right)\right] \tag{B.4}$$

By using (B.4) and the $\tilde{c}_n$ given by the computer one can prove, by trivial computations, that for any $n \leqslant 10$ $|\epsilon_n| < 2^7u$; the same bound holds for any n, as one can prove by recurrence by using in (B.4) the bounds given in Appendix A. Hence we have $\forall n$, $|\epsilon_n| < \epsilon = 2^{-40}$.

Now we shall estimate the computational error in computing the polynomial $\eta_1^{20}(t)$. It is computed by the recurrence relation

$$\eta_1^0 = \tilde{c}_{20}^{(1)}, \qquad \eta_1^k = t\eta_1^{k-1} + \tilde{c}_{20-k}^{(1)}$$

Hence $\eta_1^k(t)$ is represented in the computer by a number $\tilde{\eta}_1^k(t)$ such that

$$\tilde{\eta}_1^k(t) = (1+u_1)\left[\tilde{c}_{20-k}^{(1)} + (1+u_2)(1+u_3)t\tilde{\eta}_1^{k-1}(t)\right] \tag{B.5}$$

where $|u_i| < u$ $(i = 1,2,3)$. We have

$$\tilde{\eta}_1^k(t) = (1+u_1)\left(c_{20-k}^{(1)} + \epsilon_{20-k}\right)$$
$$+ (1+u_1)(1+u_2)(1+u_3)t\left[\eta_1^{k-1}(t) + \tilde{\eta}_1^{k-1}(t) - \eta_1^{k-1}(t)\right]$$

Since we are interested in values of t such that $|(1+u_1)(1+u_2)(1+u_2)t| < 1$, we get

$$|\tilde{\eta}_1^k(t) - \eta_1^k(t)| < \epsilon(1+u) + u|c_{20-k}^{(1)}| + |\tilde{\eta}_1^{k-1}(t) - \eta_1^{k-1}(t)| + 4u|\eta_1^{k-1}(t)|$$

$$|\tilde{\eta}_1^{20}(t) - \eta_1^{20}(t)| < 20\epsilon(1+u) + u\sum_{k=1}^{20}|c_k^{(1)}| + 4u\sum_{k=1}^{20}|\eta_1^{k-1}(t)|$$
$$< 21\epsilon + u\sum_{k=1}^{20}(4k+1)|c_k^{(1)}|$$

The values given by the computer show that

$$\sum_{k=1}^{20}|c_k^{(1)}|(4k+1) < 2^7 = \frac{\epsilon}{u}$$

hence we have

$$|\tilde{\eta}_1^{20}(t) - \eta_1^{20}(t)| < 22\epsilon < 10^{-10}$$

Similar estimates hold for the other polynomials considered in Section 3.

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